Advanced Computer Graphics
Parameterization

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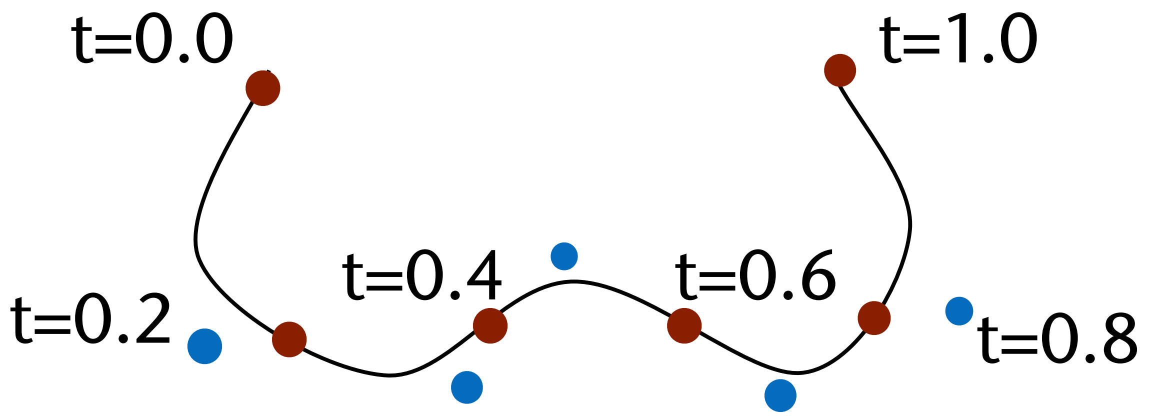
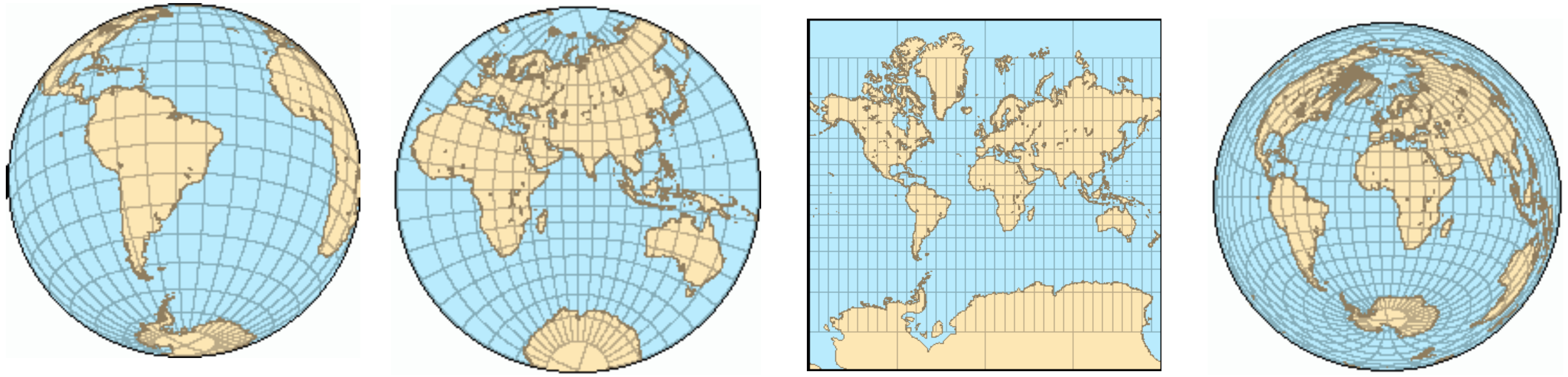
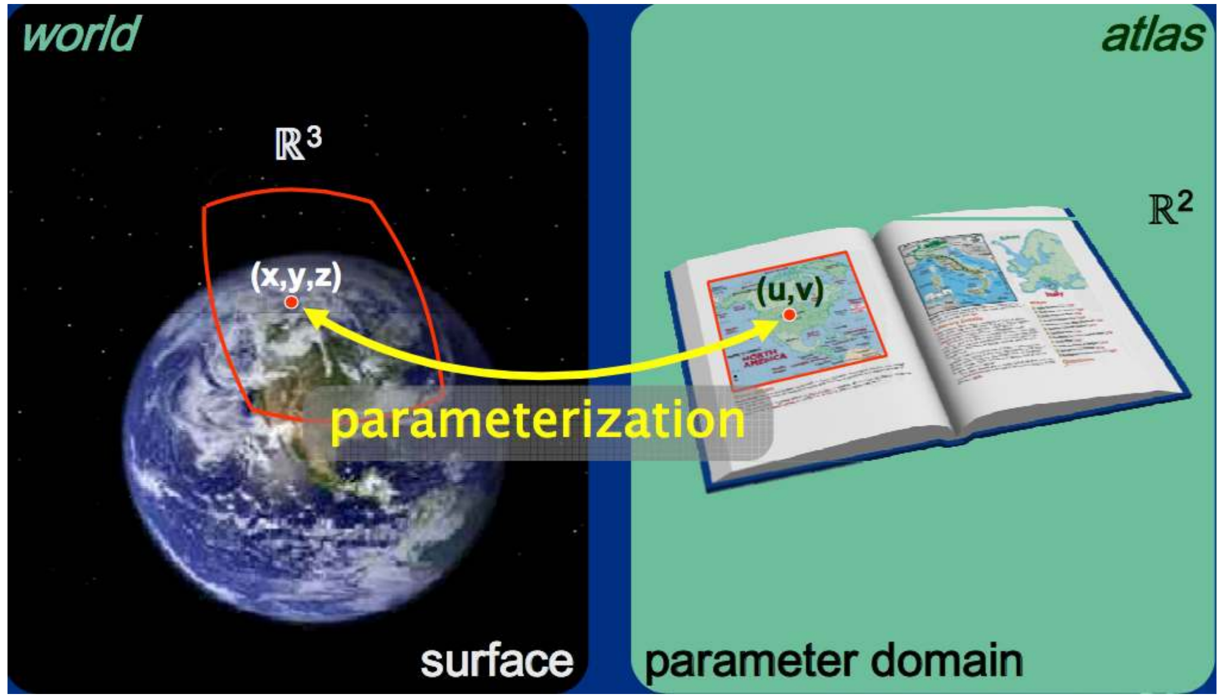
Examples of Parameterization

- The line parameter *t* on a straight line
- The knot vector of B-splines
- Latitude/longitude coordinates on the globe

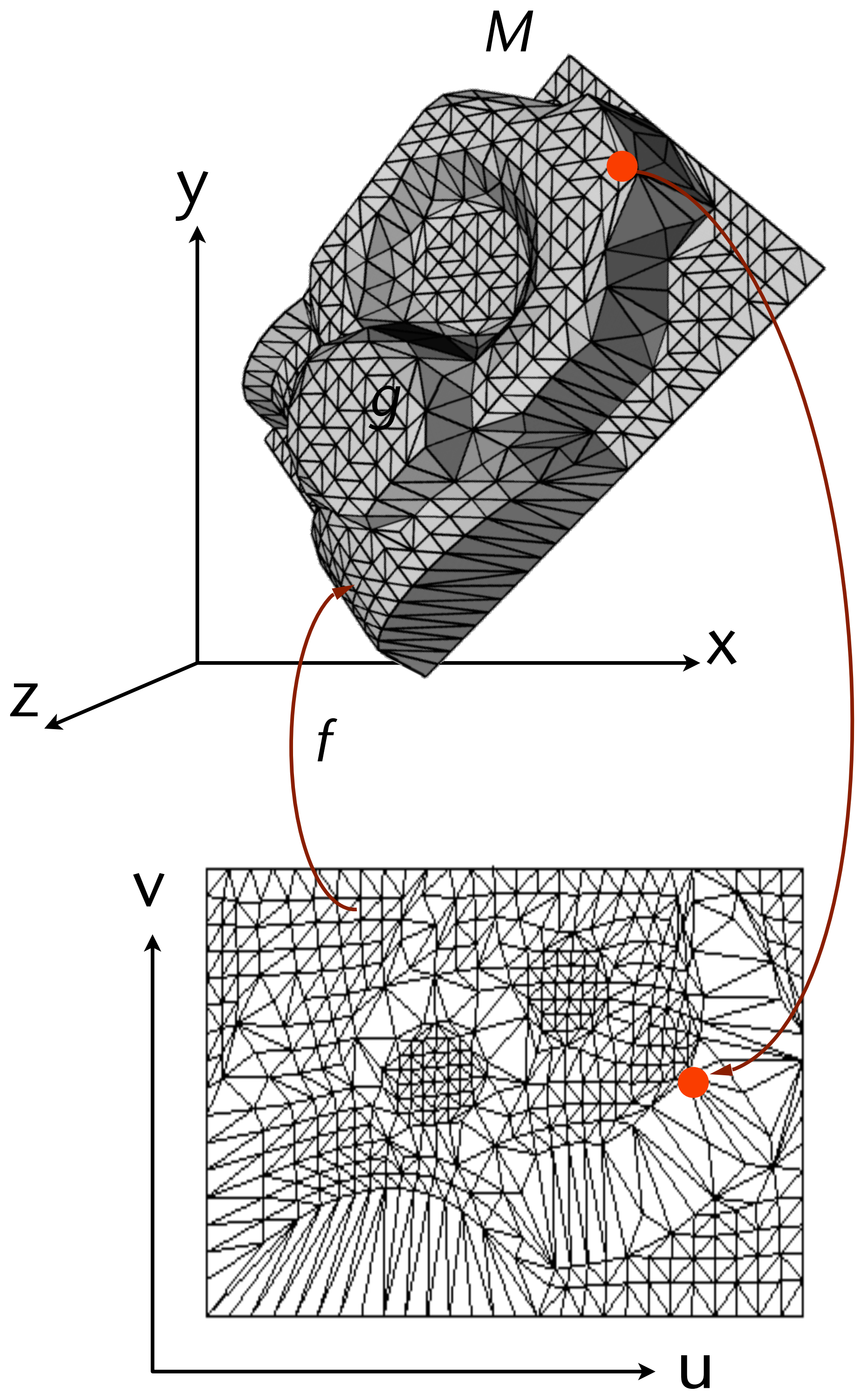






Notation and Terms

- Problem definition:
  Let $V = \{P_1, \ldots, P_N\} \subset \mathbb{R}^3$ be the set of vertices of a mesh M.
  Find a mapping (= parameterization)
  $$g : V \rightarrow \mathbb{R}^2$$
  with the following properties:
  - $g(M)$ must not contain self-intersections
    - I.e., no *inverted triangles*
  - Otherwise, $f = g^{-1}$ would not exist!
- Using barycentric interpolation, the function g can be extended to the interior of the triangles

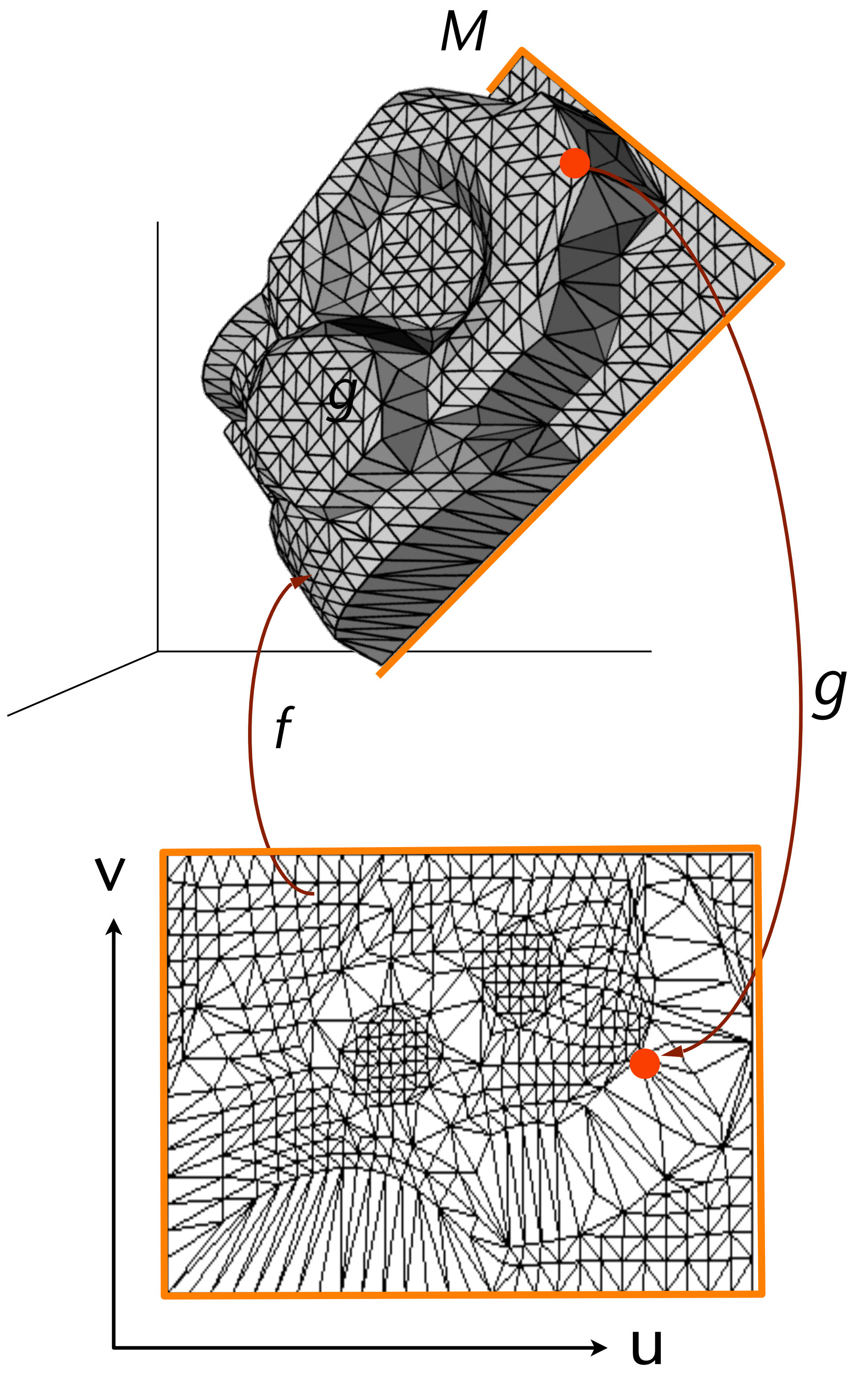


More notation

- P_i = mesh vertices, p_i = parameter points
- $V = V_I \cup V_B$

  $V_I = \{P_1, \ldots, P_n\}$ = "inner" vertices

  $V_B = \{P_{n+1}, \ldots, P_{n+b}\}$ = "boundary" vertices
- $N = n + b$
- $p_{n+1}, \ldots, p_{n+b}$ = boundary polygon in the parameter domain u,v
- $g(P_i) = p_i = (u_i, v_i)$
- E = set of edges, *corresponding in M and in g(M)*



Motivation of the Parameterization Method

- Fix the border polygon $p_{n+1}, \ldots, p_{n+b}$
- How to determine the interior p_i's ?
- Idea: "edges = springs"
  - Assumption: rest length of springs = 0
  - So, energy stored in an extended spring = $\frac{1}{2} D s^2$
    where D = spring constant, s = length of the spring
  - Set $D_{ij} > 0$ for all edges (p_i, p_j) , and set $D_{ij} = 0$ for all other (i,j)
  - Generalization: we allow $D_{ij} \neq D_{ji}$!
- Define the total energy of a parameterization: $E = \sum_{i=1}^{n} \sum_{j=1}^{N} D_{ij} \|p_i - p_j\|^2$
- Goal: minimize this energy (penalty function)

The Parameterization Method

- Partial derivatives of E are

$$\forall i = 1 \ldots n: \quad \frac{\partial E}{\partial p_i} = \sum_{j=1}^{N} D_{ij}(p_i - p_j)$$

- Setting those to 0 yields

$$\forall i = 1 \ldots n: \quad p_i \sum_{j=1}^{N} D_{ij} = \sum_{j=1}^{N} D_{ij} p_j$$

- In other words: each interior parameter point p_i must be a convex combination of its neighbors (its 1-ring), in particular

$$\forall i = 1 \ldots n: \quad p_i = \sum_{j=1}^{N} \lambda_{ij} p_j \ , \quad \text{mit} \quad \lambda_{ij} = \frac{D_{ij}}{\sum_{k=1}^{N} D_{ik}}$$

- Splitting the sum on the right hand side yields

$$p_i = \sum_{j=1}^{n} \lambda_{ij} p_j + \sum_{j=n+1}^{N} \lambda_{ij} p_j$$

and thus

$$p_i - \sum_{j=1}^{n} \lambda_{ij} p_j = \sum_{j=n+1}^{N} \lambda_{ij} p_j$$

- These are two simple linear equation systems $A\mathbf{u} = \mathbf{b} \quad \text{und} \quad A\mathbf{v} = \mathbf{c}$ (1)

where $A = (a_{ij})_{n\times n} \quad \mathbf{u} = (u_1, \ldots, u_n) \quad \mathbf{v} = (v_1, \ldots, v_n)$

with $a_{ij} = \begin{cases} 1 & , i = j \\ -\lambda_{ij} & , (p_i, p_j) \in E \\ 0 & , \text{ sonst} \end{cases} \quad , \quad b_i = \sum_{j=n+1}^{N} \lambda_{ij} u_j \quad , \quad c_i = \sum_{j=n+1}^{N} \lambda_{ij} v_j$

- Final step for generating the parameterization: choose λ's such, that

$$\forall (i,j) \in E: \; \lambda_{ij} > 0 \quad , \quad \forall (i,j) \notin E: \; \lambda_{ij} = 0 \quad , \quad \sum_{j=1}^{N} \lambda_{ij} = 1$$

$$i = 1 \ldots n \, , \quad j = 1 \ldots N$$

then solve the LES for **u** and **v**

- Theorem:
  If the λ's are chosen as described above, then the matrix A is non-singular.

- In other words: The linear systems have a unique solution

Proof

- Definition:
  An *nxn* matrix *A* is called decomposable (aka reducible) ⇔ there exists a permutation matrix *P* such that

$$A' = P^{-1}AP = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$$

  where *B* and *D* are square matrices, too.
  Otherwise it is called non-decomposable.

- Note: In our application, *P* is equivalent to a renumbering of the vertices in *M* and, likewise, the parameter points

- In our case: A is of the special form

$$A = I - \Lambda$$

where

$$\Lambda = (\lambda_{ij}),\ i,j = 1 \ldots n, \quad \lambda_{ij} \geq 0$$

- Conjecture: Λ is non-decomposable (thus, A is non-decomposable, too)
- Proof:

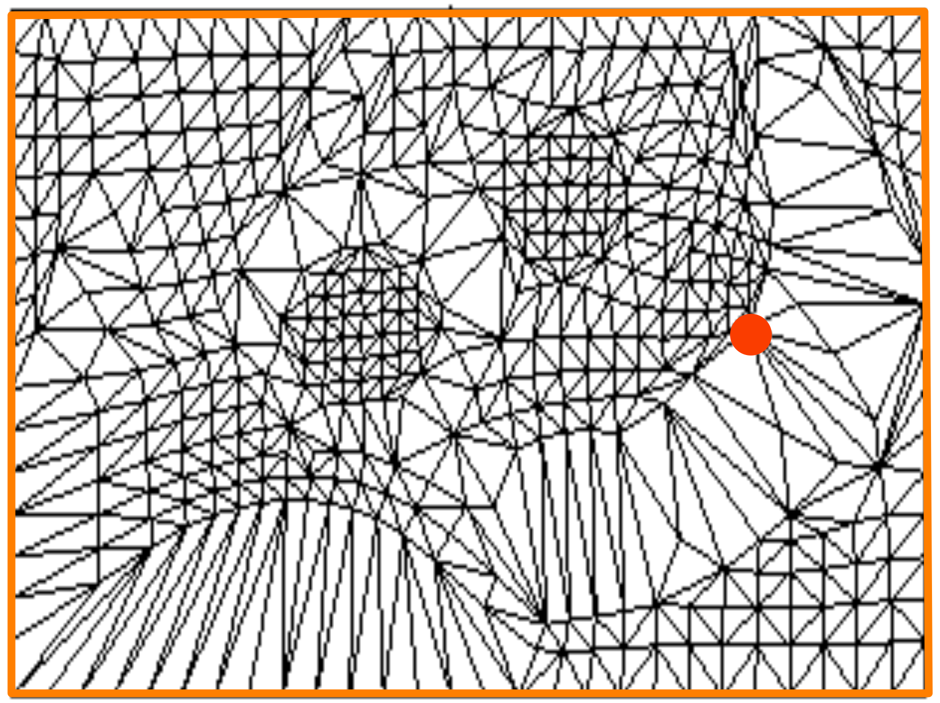
1. If Λ was decomposable, then a renumbering of vertices would be possible such that

$$\Lambda = \begin{pmatrix} B & 0 \\ 0 & D \end{pmatrix}$$

   Note: $\lambda_{ij} = 0 \Leftrightarrow \lambda_{ji} = 0$

2. Consequence: the graph (mesh) would consist of 2 non-connected parts ⚡

- Further notes on matrix Λ:
  - Every row i corresponds to the inner point p_i
  - $\lambda_{ij} > 0 \Leftrightarrow (i,j) \in E$
  - Note: Λ does *not* contain λ_{ij}'s corresponding to edges connecting a boundary point!
  - If p_i has no edges to boundary points, then $\sum_{j=1}^{n} \lambda_{ij} = 1$
  - If p_i does have edges to boundary points, then $\sum_{j=1}^{n} \lambda_{ij} < 1$



- Theorem from matrix theory (without proof):

  Let A be a non-decomposable matrix with non-negative elements.

  Denote the sums of the rows with

$$s_i = \sum_{j=1}^{n} a_{ij} \ , \quad i = 1 \ldots n$$

  Assume that A has the property that

$$\min_{i=1 \ldots n} s_i \lneq \max_{i=1 \ldots n} s_i$$

  Let r be the *maximum* eigenvalue of A.

  Then, $r < \max_{i=1 \ldots n} s_i$.

- Now for the proof that A is *non-singular*:
  - We have to show
$$Aw = 0 \;\Leftrightarrow\; w = 0$$
  - Plugging in yields
$$(I - \Lambda)w = 0 \;\Leftrightarrow\; \Lambda w = w$$
  - Assumption: there exists such a $w \neq 0$
  - Then, 1 would be an eigenvalue of Λ
  - For our Λ, we know that some of the $s_i = 1$, and some $s_i < 1$
  - Therefore, by the previous theorem: the maximal eigenvalue < 1 → ⚡

Some Concrete Choices for the λ's

- Naïve choice [1963, graph drawing]:
  - Set $\lambda_{ij} = 1/d_i$ for each P_i , where d_i = degree of the vertex = #neighbors
  - In other words: Each p_i is the "center of mass" of its neighbors
  - This is called uniform parameterization
    - By analogy to uniform parameterization for B-splines

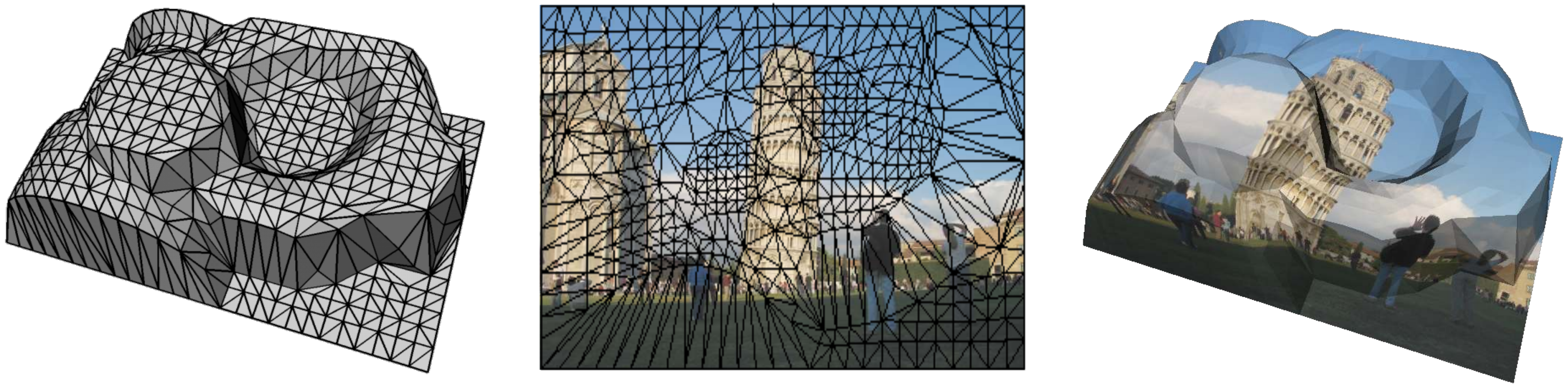
- Chord length parameterization:
  - Set $w_{ij} = 1/\|P_j - P_i\|$ (in 3D space)
  - Set $\lambda_{ij} = \frac{w_{ij}}{\sum_{j=1}^{N} w_{ij}}$
  - So, the stiffness of edges in the 2D mesh (in the parameter domain) is inversely proportional to the length of their edges in the 3D mesh

- Use mean value coordinates (MVC):
  - Set the λ_{ij} = the mean value coordinates of P_i with respect to its direct neighbors P_j in the 3D mesh *M* (!)
  - One version how to do this:
    - Determine for each P_i its direct neighbors P_j (= 1-ring of P_i)
    - Determine a least squares plane through these points (linear regression)
    - Project these points onto that plane
    - Determine the mean value coordinates of P_i w.r.t. P_j in that plane (now this is a 2D MVC problem)
- Now it is clear, why we had to allow $\lambda_{ij} \neq \lambda_{ji}$!

Putting it All Together

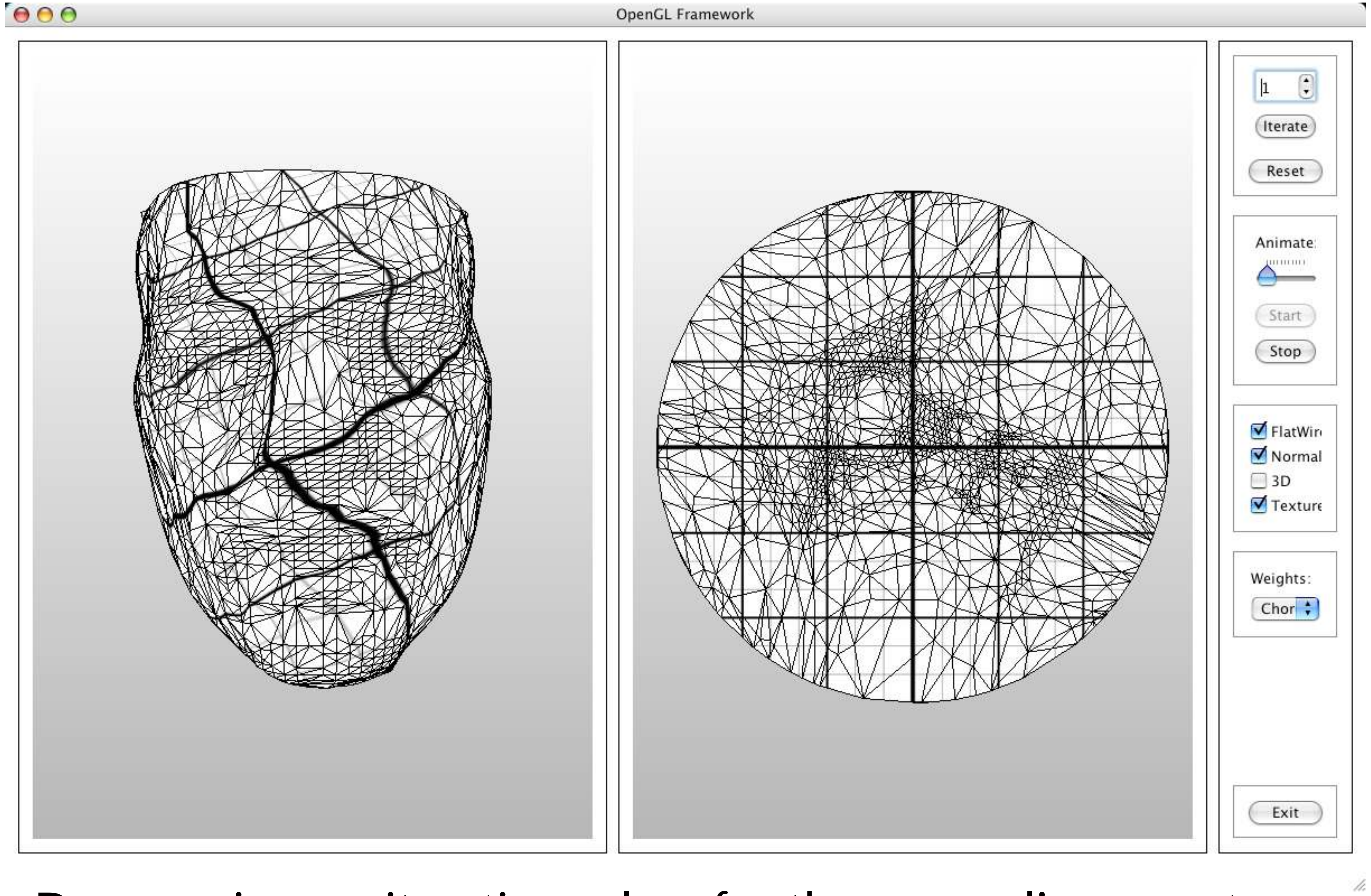
- Calculate all the λ_{ij}
- Having those, set up matrix A and vectors $\mathbf{b}$ and $\mathbf{c}$
- Solve the LES's $A\mathbf{u} = \mathbf{b} \quad \text{und} \quad A\mathbf{v} = \mathbf{c}$
  - Use a sparse solver to solve for **u** and **v**

Application of the Parameterization to Texturing



Many further applications of such parameterization methods exist, because parameterization allows us to operate on a mesh, as if it was flat, i.e., living in the 2D plane.

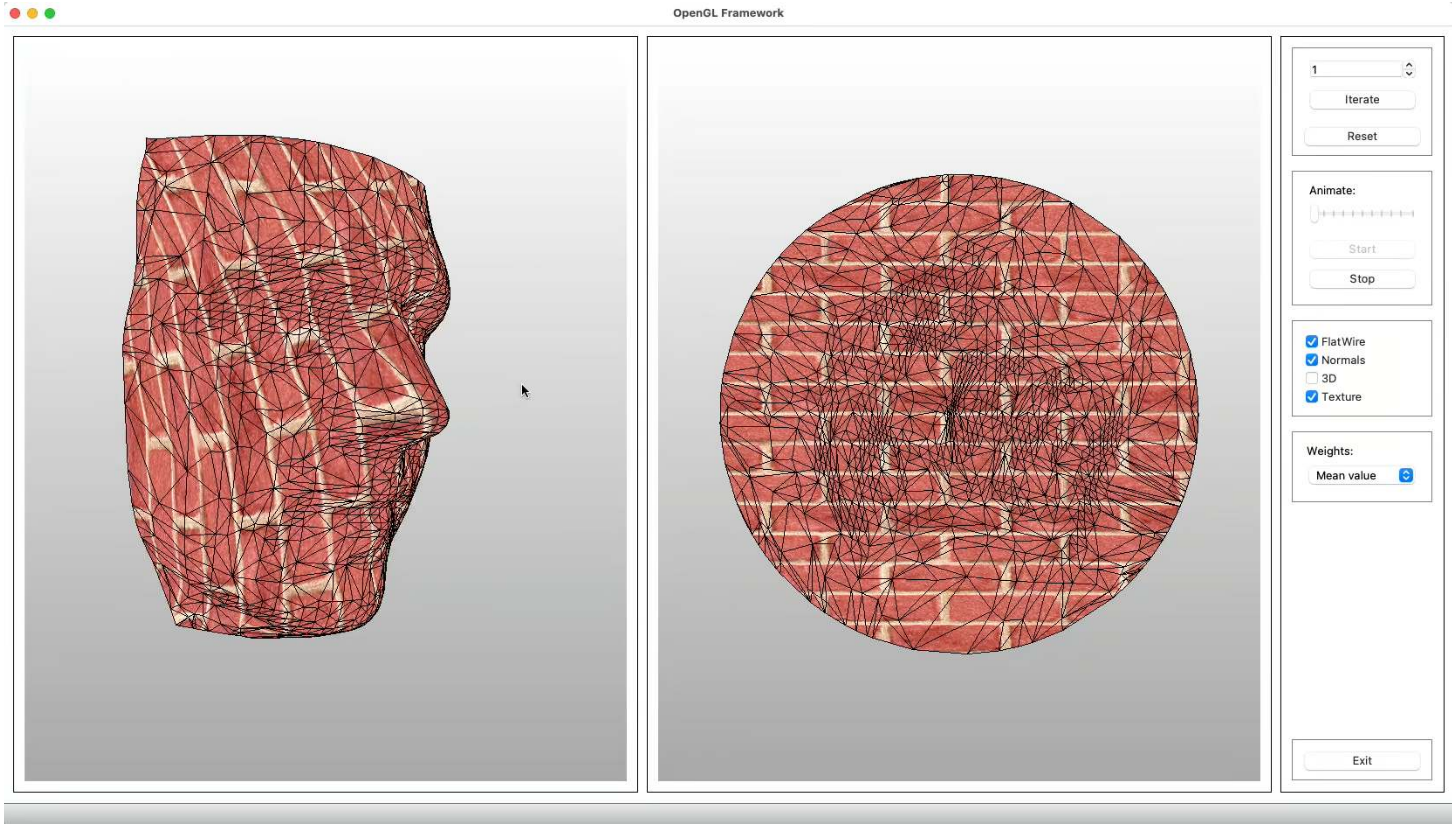
Demo

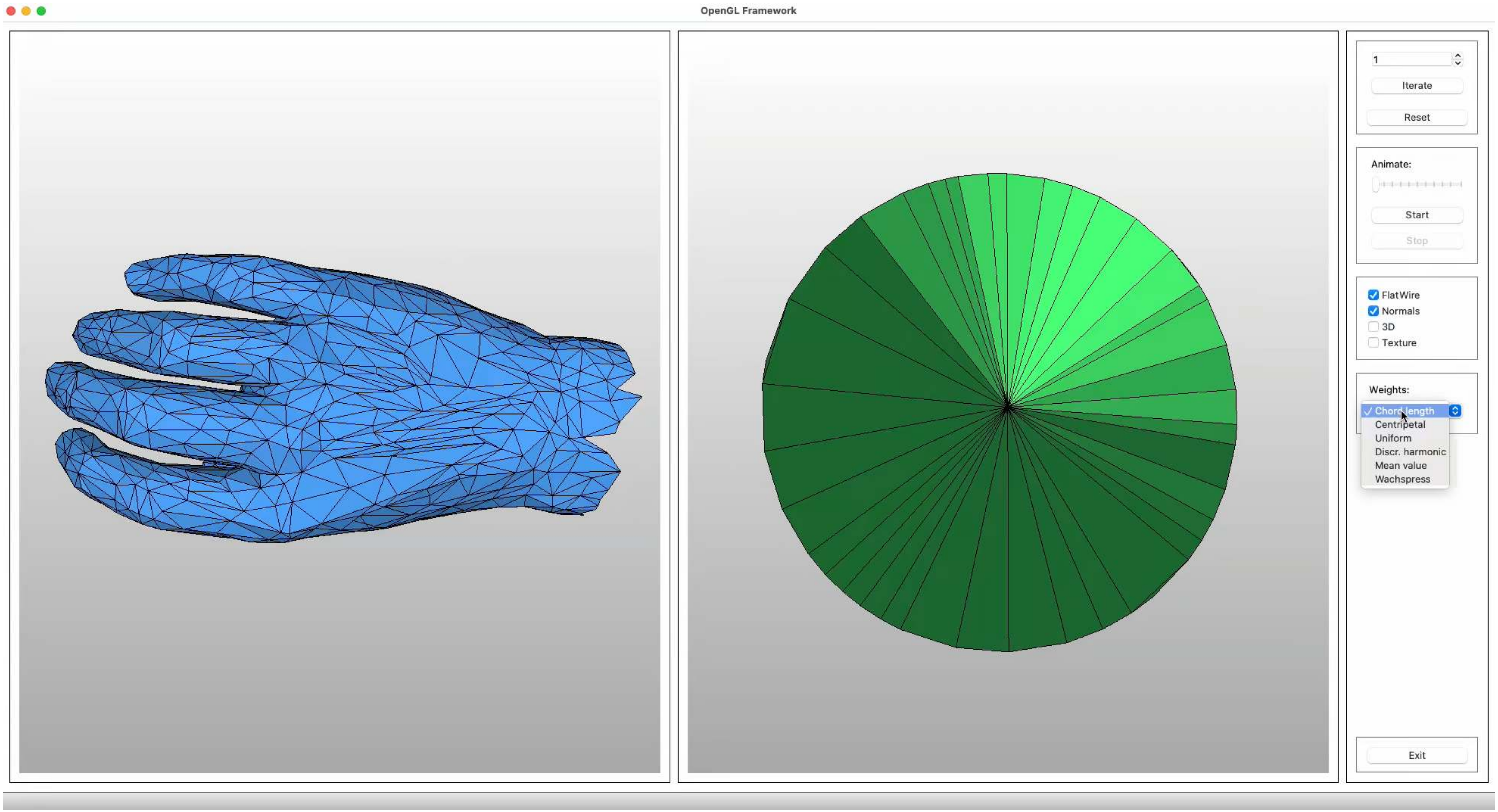


Demo using an iterative solver for the sparse linear system, showing intermediate solutions of the parameterization

Bachelor-Arbeit / Indep. Study: bessere Demo, die mehr OBJ's verarbeiten kann; evtl. Zusätzliche Aufgabe: besseres boundary handling (s. Siggraph-Tutorial)

Videos of the Demo




OpenGL Framework
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Animate:
Start
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FlatWire
Normals
3D
Texture
Weights:
Chord length
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Mean value
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